

# Blow-ups of locally conformally Kähler manifolds

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## Abstract

A locally conformally Kähler (LCK) manifold is a manifold which is covered by a Kähler manifold, with the deck transform group acting by homotheties. We show that the blow-up of a compact LCK manifold along a complex submanifold admits an LCK structure if and only if this submanifold is globally conformally Kähler. We also prove that a twistor space (of a compact 4-manifold, a quaternion-Kähler manifold or a Riemannian manifold) cannot admit an LCK metric, unless it is Kähler.

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# 1 Introduction

## 1.1 Bimeromorphic maps and locally conformally Kähler structures

A **locally conformally Kähler** (LCK) manifold is a complex manifold  $M$ ,  $\dim_{\mathbb{C}} M > 1$ , admitting a Kähler covering  $(\tilde{M}, \tilde{\omega})$ , with the deck transform group acting on  $(\tilde{M}, \tilde{\omega})$  by holomorphic homotheties. Unless otherwise stated, we shall consider only compact LCK manifolds.

In the present paper we are interested in the birational (or, more precisely, bimeromorphic) geometry of LCK manifolds.

An obvious question arises immediately.

**Question 1.1:** Let  $X \subset M$  be a complex subvariety of an LCK manifold, and  $M_1 \rightarrow M$  a blowup of  $M$  in  $X$ . Would  $M_1$  also admit an LCK structure?

When  $X$  is a point, the question is answered in affirmative by Tricerri [Tr] and Vuletescu [Vu]. When  $\dim X > 0$ , the answer is not immediate. To state it properly, we recall the notion of a **weight bundle** of an LCK manifold. Let  $(\tilde{M}, \tilde{\omega})$  be the Kähler covering of an LCK manifold  $M$ , and  $\pi_1(M) \rightarrow \text{Map}(\tilde{M}, \tilde{M})$  the deck transform map. Since  $\rho^*(\gamma)\tilde{\omega} = \text{const} \cdot \tilde{\omega}$ , this constant defines a character  $\pi_1(M) \xrightarrow{\chi} \mathbb{R}^{>0}$ , with  $\chi(\gamma) := \frac{\rho^*(\gamma)\tilde{\omega}}{\tilde{\omega}}$ .

**Definition 1.2:** Let  $L$  be the 1-dimensional local system on  $M$  with monodromy defined by the character  $\chi$ . We think of  $L$  as of a real bundle with a flat connection. This bundle is called **the weight bundle** of  $M$ .

One may think of the Kähler form  $\tilde{\omega}$  as of an  $L$ -valued differential form on  $M$ . This form is closed, positive, and of type (1,1). Therefore, for any smooth complex subvariety  $Z \subset M$  such that  $L|_Z$  is a trivial local system,  $Z$  is Kähler.

The following two theorems describe how the LCK property behaves under blow-ups.

**Theorem 1.3:** Let  $Z \subset M$  be a compact complex submanifold of an LCK manifold, and  $M_1$  the blow-up of  $M$  with center in  $Z$ . If the restriction  $L|_Z$  of the weight bundle is trivial as a local system then  $M_1$  admits an LCK

metric.

**Proof:** See Corollary 2.11. ■

A similar question about blow-downs is also answered.

**Theorem 1.4:** Let  $D \subset M_1$  be an exceptional divisor on an LCK manifold, and  $M$  the complex variety obtained as a contraction of  $D$ . Then the restriction  $L|_D$  of the weight bundle to  $D$  is trivial.

**Proof:** Theorem 2.9. ■

This result is quite unexpected, and leads to the following theorem about a special class of LCK manifold called *Vaisman manifolds* (Section 2).

**Claim 1.5:** Let  $M$  be a Vaisman manifold. Then any bimeromorphic contraction  $M \rightarrow M'$  is trivial. Moreover, for any positive-dimensional submanifold  $Z \subset M$ , its blow-up  $M_1$  does not admit an LCK structure.

**Proof:** Corollary 2.13 ■

## 1.2 Positive currents on LCK manifolds

The proofs of Theorem 1.4 and Theorem 1.3 are purely topological. However, they were originally obtained using a less elementary argument involving positive currents.

We state this argument here, omitting minor details of the proof, because we think that this line of thought could be fruitful in other contexts too; for more information and missing details, the reader is referred to [D1], [DP] and [D2].

A **current** is a form taking values in distributions. The space of  $(p, q)$ -currents on  $M$  is denoted by  $D^{p,q}(M)$ . A **strongly positive current**<sup>1</sup> is a linear combination

$$\sum_I \alpha_I (z \wedge \bar{z})_I$$

where  $\alpha_I$  are positive, measurable functions, and the sum is taken over all multi-indices  $I$ . An integration current of a closed complex subvariety is a strongly positive current.

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<sup>1</sup>In the present paper, we shall often omit “strongly”, because we are only interested in strong positivity.

It is easy to define the de Rham differential on currents, and check that its cohomology coincide with the de Rham cohomology of the manifold.

Currents are naturally dual to differential forms with compact support. This allows one to define an integration (pushforward) map of currents, dual to the pullback of differential forms. This map is denoted by  $\pi_*$ , where  $\pi : M \rightarrow N$  is a proper morphism of smooth manifolds.

Now, let  $\pi : M \rightarrow N$  be a blow-up of a subvariety  $Z \subset N$  of codimension  $k$ , and  $\omega$  a Kähler form on  $M$ . Then  $(\pi_*\omega)^k$  has a singular part which is proportional to the integration current of  $Z$ .

This follows from the Siu's decomposition of positive currents ([D1]). Demailly's results on intersection theory of positive currents ([D2]) are used to multiply the currents, and the rest follows because the Lelong numbers of  $\pi_*\omega$  along  $Z$  are non-zero.

Applying this argument to a birational contraction  $M \xrightarrow{\varphi} M'$  of an LCK manifold  $M$ , and denoting by  $\tilde{M} \xrightarrow{\tilde{\varphi}} \tilde{M}'$  the corresponding map of coverings, we obtain a closed, positive current  $\xi := \tilde{\varphi}_*\tilde{\omega}$  on  $\tilde{M}'$ , with the deck transform map  $\rho$  acting on  $\xi$  by homotheties. Then  $\rho$  would also act by homotheties on the current  $\xi^k$ ,  $k = \dim Z$ , where  $Z$  is the exceptional set of  $\tilde{\varphi}$ .

Applying the above result to decompose  $\xi^k$  onto its absolutely continuous and singular part, we obtain that the current of integration  $[Z]$  of  $Z$  is mapped to  $\text{const}[Z]$  by the deck transform action. Since the current of integration of  $Z$  is mapped by the deck transform to the current of integration of  $\tilde{\varphi}(Z) = Z$ , the constant  $\text{const}$  is trivial; this implies that  $\pi(Z) \subset M'$  is Kähler, with the Kähler metric obtained in the usual way from  $\tilde{\omega}$ .

### 1.3 Fujiki class C and LCK geometry

A compact complex variety  $X$  is said to belong to **Fujiki class C** if  $X$  is bimeromorphic to a Kähler manifold. The Fujiki class C manifolds are closed under many natural operations, such as taking a subvariety, or the moduli of subvarieties, and play important role in Kähler geometry.

This notion has a straightforward LCK analogue.

**Definition 1.6:** Let  $M$  be a compact complex variety. It is called a **locally conformally class C variety** if it is bimeromorphic to an LCK manifold.

The importance of the Fujiki class C notion was emphasized by a more recent work of Demailly and Păun [DP], who characterized class C manifolds in terms of positive currents. Recall that a **Kähler current** is a positive,

closed (1,1)-current  $\varphi$  on a complex manifold  $M$  which satisfies  $\varphi \geq \omega$  for some Hermitian form  $\omega$  on  $M$ .

Demailly and Păun have proven that a compact complex manifold  $M$  belongs to class C if and only if it admits a positive Kähler current.

For an LCK manifold, an analogue of a Kähler current is provided by the following notion (motivated by Definition 2.2).

**Definition 1.7:** Let  $M$  be a compact complex manifold,  $\theta$  a closed real 1-form on  $M$ ,  $\Xi$  a positive, real (1,1)-current satisfying  $d\Xi = \theta \wedge \Xi$  and  $\Xi \geq \omega$  for some Hermitian form  $\omega$  on  $M$ . Then  $\Xi$  is called **an LCK current**.

It would be interesting to know if an LCK-analogue of the Demailly-Păun theorem is true.

**Question 1.8:** Let  $M$  be a complex compact manifold. Determine whether the following conditions are equivalent.

- (i)  $M$  belongs to locally conformally class C.
- (ii)  $M$  admits an LCK current.

## 2 Blow-ups and blow-downs of LCK manifolds

We start by repeating (in a more technical fashion) the definition of an LCK manifold given in the introduction. Please see [DO] for more details and several other versions of the same definition, all of them equivalent.

**Definition 2.1:** A **locally conformally Kähler** (LCK) manifold is a complex manifold  $X$  covered by a system of open subsets  $U_\alpha$  endowed with *local* Kähler metrics  $g_\alpha$ , conformal on overlaps  $U_\alpha \cap U_\beta$ :  $g_\alpha = c_{\alpha\beta} g_\beta$ .

Note that, in complex dimension at least 2, as we always assume,  $c_{\alpha\beta}$  are positive constants. Moreover, they obviously satisfy the cocycle condition. Interpreted in cohomology, the cocycle  $\{c_{\alpha\beta}\}$  determines a closed one-form  $\theta$ , called **the Lee form**. Hence, locally  $\theta = df_\alpha$ . It is easily seen that  $e^{-f_\alpha} g_\alpha = e^{-f_\beta} g_\beta$  on  $U_\alpha \cap U_\beta$ , and thus determine a *global* metric  $g$  which is conformal on each  $U_\alpha$  with a Kähler metric. One obtains the following equivalent:

**Definition 2.2:** A Hermitian manifold  $M$  is LCK if its fundamental two-form  $\omega$  satisfies:

$$d\omega = \theta \wedge \omega, \quad d\theta = 0. \quad (2.1)$$

for a *closed* one-form  $\theta$ .

If  $\theta$  is exact then  $M$  is called **globally conformally Kähler** (GCK).

As we work with compact manifolds and, in general, the topology of compact Kähler manifolds is very different from the one of compact LCK manifolds, we always assume  $\theta \neq 0$  on  $X$ .

Let  $\Gamma \longrightarrow \tilde{M} \xrightarrow{\pi} M$  be the universal cover of  $M$  with deck group  $\Gamma$ . As  $\pi^*\theta$  is exact on  $\tilde{M}$ ,  $\pi^*\omega$  is globally conformal with a Kähler metric  $\tilde{\omega}$ . Moreover,  $\Gamma$  acts by holomorphic homotheties with respect to  $\tilde{\omega}$ . This defines a character

$$\chi : \Gamma \longrightarrow \mathbb{R}^{>0}, \quad \gamma^*\tilde{\omega} = \chi(\gamma)\tilde{\omega}. \quad (2.2)$$

It can be shown that this property is indeed an equivalent definition of LCK manifolds, see [OV2].

Clearly, a LCK manifold  $M$  is globally conformally Kähler if and only if  $\Gamma$  acts trivially on  $\tilde{\omega}$  (*i.e.*  $\text{im } \chi = \{1\}$ ).

A particular class of LCK manifolds are the **Vaisman manifolds**. They are LCK manifolds with the Lee form parallel with respect to the Levi-Civita connection of the LCK metric. The compact ones are mapping tori over the circle with Sasakian fibre, see [OV1]. The typical example is the Hopf manifold, diffeomorphic to  $S^1 \times S^{2n-1}$ .

On a Vaisman manifold, the vector field  $\theta^\# - \sqrt{-1}J\theta^\#$  generates a one-dimensional holomorphic, Riemannian, totally geodesic foliation. If this is regular and if  $M$  is compact, then the leaf space  $B$  is a Kähler manifold.

**Example 2.3:** On a Hopf manifold  $\mathbb{C}^n \setminus \{0\} / \langle z_i \mapsto 2z_i \rangle$ , the LCK metric  $\frac{\sum dz_i \otimes dz_i}{|\sum z_i \bar{z}_i|^2}$  is Vaisman and regular; the leaf space is  $\mathbb{C}P^{n-1}$ .

We refer to [DO] or to the more recent [OV2] for more details about LCK geometry.

It is known, [Tr, Vu], that the blow-up **at points** preserve the LCK class. The present paper is devoted to the blow-up of LCK manifolds along

subvarieties. In this case, the situation is a bit more complicated and a discussion should be made according to the dimension of the submanifold.

**Definition 2.4:** Let  $Y \xrightarrow{j} M$  be a complex subvariety. We say that  $Y$  is of **induced globally conformally Kähler type** (IGCK) if the cohomology class  $j^*[\theta]$  vanishes, where  $\theta$  denotes the cohomology class of the Lee form on  $M$ .

**Remark 2.5:** Notice that a IGCK-submanifold of an LCK manifold is always Kähler.

**Remark 2.6:** By a theorem of Vaisman ([Va2]), any LCK metric on a compact complex manifold  $Y$  of Kähler type is globally conformally Kähler if  $\dim_{\mathbb{C}} Y > 1$ . Therefore, the IGCK condition above for smooth  $Y$  with  $\dim_{\mathbb{C}} Y > 1$  is equivalent to  $Y$  being Kähler.

**Remark 2.7:** Notice that there may exist curves on LCK manifolds which are not IGCK, despite being obviously of Kähler type. For instance, if  $M$  is a regular Vaisman manifold, and if  $Y$  is a fiber of its elliptic fibration, then  $Y$  is not IGCK, as any compact complex subvariety of a compact Vaisman manifold has an induced Vaisman structure (see *e.g.* [Ve1, Proposition 6.5]).

The main goal of the present paper is to prove the following two theorems:

**Theorem 2.8:** Let  $M$  be an LCK manifold,  $Y \subset M$  be a smooth complex IGCK subvariety, and let  $\tilde{M}$  be the blow-up of  $M$  centered in  $Y$ . Then  $\tilde{M}$  is LCK.

**Proof:** See the argument after Lemma 3.4. ■

**Theorem 2.9:** Let  $M$  be a complex variety, and  $\tilde{M} \rightarrow M$  the blow-up of a compact subvariety  $Y \subset M$ . Assume that  $\tilde{M}$  is smooth and admits an LCK metric. Then the blow-up divisor  $\tilde{Y} \subset \tilde{M}$  is a IGCK subvariety.

**Proof:** See Remark 3.3. ■

**Remark 2.10:** In the situation described in Theorem 2.9, the variety  $\tilde{Y}$  is of Kähler type, because it is IGCK. When  $Y$  is smooth,  $Y$  is Kähler, as

shown by Blanchard ([Bl2, Théorème II.6]). Together with Remark 2.6, this implies the following corollary.

**Corollary 2.11:** Let  $M$  be an LCK manifold, and  $Y \subset M$  a smooth compact subvariety, such that the blow-up of  $M$  in  $Y$  admits an LCK metric. If  $\dim_{\mathbb{C}}(Y) > 1$  then  $Y$  is a IGCK subvariety. ■

**Remark 2.12:** Note that, from [Ve1, Proposition 6.5], a compact complex submanifold  $Y$  of a compact Vaisman manifold is itself Vaisman, and  $\theta$  represents a non-trivial class in the cohomology of  $Y$ , so there are no IGCK submanifolds of proper dimension  $\dim_{\mathbb{C}}(Y) > 0$ . This implies the following corollary.

**Corollary 2.13:** The blow-up of a compact Vaisman manifold along a compact complex submanifold  $Y$  of dimension at least 1 cannot have an LCK metric.

The proofs of these two theorems and of the corollary will be given in Section 3. As a by-product of our proof, we obtain the following:

**Corollary 2.14:** If  $M$  is a twistor space, and if  $M$  admits a LCK metric, then this metric is actually GCK.

**Proof:** See Corollary 3.2. ■

Here, by a “twistor space” we understand any of the following constructions of a complex manifold: the twistor spaces of half-conformally flat 4-dimensional Riemannian manifolds, twistor spaces of quaternionic-Kähler manifolds, and Riemannian twistor spaces of conformally flat manifolds.

**Remark 2.15:** (i) A similar, weaker result is proven in [KK]. Namely, the twistor space of half-conformally flat 4-dimensional Riemannian manifolds with large fundamental group cannot admit LCK metrics with automorphic potential on the covering. The proof uses different techniques from ours, and which cannot be generalized neither to higher dimensions nor to quaternionic Kähler manifolds.

(ii) It was known from [Ga, Mu] that the natural metrics (with respect to the twistor submersion) cannot be LCK. Our result refers to any metric on the twistor space, not necessarily related to the twistor submersion. On the other hand, as shown by Hitchin, the twistor space of a compact 4-dimensional manifold is not of Kähler type, unless it is biholomorphic to



$\mathbb{C}P^3$  or to the flag variety  $F_2$  [Hi].

**Remark 2.16:** So far, we were unable to deal with the reverse statement of Theorem 2.8, namely, to determine whether a smooth bimeromorphic contraction of an LCK manifold is always LCK. In the particular case when an exceptional divisor is contracted to a point, this has been proven to be true by Tricerri, [Tr]; we conjecture that in the general case this is false, but we are not able to find any example.

For GCK (that is, Kähler) manifolds, the answer is well known: blow-downs of Kähler manifolds can be non-Kähler, as one can see from any example of a Moishezon manifold.

**Remark 2.17:** We summarize the case of blow-up of curves on LCK manifolds. Since rational curves are simply-connected, they are IGCK submanifolds, so blowing-up a rational curve on a LCK manifold always yields a manifold of LCK type. The case of the elliptic curves was partially tackled in Corollary 2.13. If  $Y$  is a curve of arbitrary genus contained in an exceptional divisor of a blow-up, then it is also automatically a IGCK subvariety since the exceptional divisor is so; hence again, blowing it up yields a manifold of LCK type.

To our present knowledge, the only examples of curves  $Y$  on LCK manifolds  $M$  with genus  $g(Y) \geq 2$  are curves belonging to some exceptional divisors. It would be interesting to prove that this is the case in general, or to build out a counter-example.

### 3 The proofs

**Lemma 3.1:** Let  $M$  be an LCK manifold,  $B$  a path connected topological space and let  $\pi : M \rightarrow B$  be a continuous map. Assume that either

- (i)  $B$  is an irreducible complex variety, and  $\pi$  is proper and holomorphic.
- (ii)  $\pi$  is a locally trivial fibration with fibers which are complex subvarieties of  $M$ .

Suppose also that the map

$$\pi^* : H^1(B) \rightarrow H^1(M)$$

is an isomorphism, and the generic fibers of  $\pi$  are positive-dimensional. Then the LCK structure on  $M$  is actually GCK.

**Proof:** Denote by  $\theta$  the Lee form of  $M$ , and let  $\tilde{M}$  be the minimal GCK covering of  $X$ , that is, the minimal covering  $\tilde{M} \rightarrow M$  such that the pullback of  $\theta$  is exact. Since  $H^1(B) \cong H^1(M)$ , there exists a covering  $\tilde{B} \rightarrow B$  such that the following diagram is commutative, and the fibers of  $\tilde{\pi}$  are compact:

$$\begin{array}{ccc} \tilde{M} & \longrightarrow & M \\ \tilde{\pi} \downarrow & & \downarrow \pi \\ \tilde{B} & \longrightarrow & B \end{array}$$

Let  $\tilde{B}_0 \subset \tilde{B}$  be the set of regular values of  $\tilde{\pi}$ , and let  $F_b := \tilde{\pi}^{-1}(b)$  be the regular fibers of  $\tilde{\pi}$ ,  $\dim_{\mathbb{C}} F_b = k$ . Since  $B_0$  is connected, all  $F_b$  represent the same homology class in  $H_{2k}(\tilde{M})$ .

Denote the Kähler form of  $\tilde{M}$  by  $\tilde{\omega}$ , conformally equivalent to the pullback of the Hermitian form on  $X$ .

Since all  $F_b$  represent the same homology class, the Riemannian volume

$$\text{Vol}_{\tilde{\omega}}(F_b) := \int_{F_b} \tilde{\omega}^k$$

is independent from  $b \in B_0$ . This gives (recall the definition of the character  $\chi$  in (2.2))

$$\text{Vol}_{\tilde{\omega}}(F_b) = \int_{F_b} \tilde{\omega}^k = \int_{F_{\gamma^{-1}(b)}} \gamma^* \tilde{\omega}^k = \int_{F_{\gamma^{-1}(b)}} \chi(\gamma)^k \tilde{\omega}^k = \chi(\gamma)^k \text{Vol}_{\tilde{\omega}}(F_b),$$

hence the constant  $\chi_{\gamma}$  is equal to 1 for all  $\gamma \in \Gamma$ . Therefore,  $\tilde{\omega}$  is  $\Gamma$ -invariant, and  $M$  is globally conformally Kähler. ■

The above lemma immediately implies Corollary 2.14.

**Corollary 3.2:** Let  $Z$  be the twistor space of  $M$ , understood in the sense of Corollary 2.14. Assume that  $Z$  admits an LCK metric. Then this metric is globally conformally Kähler.

**Proof:** There is a locally trivial fibration  $Z \rightarrow M$ , with complex analytic fibers which are compact symmetric Kähler spaces, hence Lemma 3.1 can be applied. ■

**Remark 3.3:** In the same way one deals with the blow-ups: the generic fibers over an exceptional set of a blow-up map are positive-dimensional. Therefore, Lemma 3.1 implies Theorem 2.9.

We can now give **The proof of Corollary 2.13:**

If  $\dim_{\mathbb{C}}(Y) > 1$  the result follows from Corollary 2.11 and Remark 2.12. In the case  $\dim_{\mathbb{C}}(Y) = 1$  we cannot use this argument directly - see Remark 2.7 - so in this case we argue as follows.

Assume  $\tilde{M}$  has an LCK metric  $\tilde{\omega}$  with Lee form  $\tilde{\eta}$ . By Theorem 2.9, the restriction  $\tilde{\eta}|_Z$  to the exceptional divisor  $Z$  is exact. Hence, after possibly making a conformal change of the LCK metric, we can assume  $\tilde{\eta}|_V = 0$  where  $V$  is a neighbourhood of  $Z$ . In particular,  $\tilde{\eta}$  will be the pull-back of a one-form  $\eta$  on  $M$ . On the other hand,  $\tilde{\omega}$  gives rise to a current on  $\tilde{M}$  (see also §1.2) and its push-forward defines an LCK positive  $(1,1)$  current  $\Xi$  on  $M$  with associate Lee form  $\eta$ . Clearly  $\eta|_Y = 0$ .

Possibly conformally changing now  $\Xi$ , we can assume that  $\eta$  is the unique harmonic form (with respect to the Vaisman metric of  $M$ ) in its cohomology class. Possibly  $\eta|_Y$  is no longer zero, but remains *exact*.

We now show that  $\eta$  is basic with respect to the canonical foliation  $\mathcal{F}$  generated on  $M$  by  $\theta^\sharp - \sqrt{-1}J\theta^\sharp$ . Indeed, from [Va2], we know that any harmonic form on a compact Vaisman manifold decomposes as a sum  $\alpha + \theta \wedge \beta$  where  $\alpha$  and  $\beta$  are basic and transversally (with respect to  $\mathcal{F}$ ) harmonic forms. In particular, as a transversally harmonic function is constant, we have

$$\eta = \alpha + c \cdot \theta, \quad (3.1)$$

where  $c \in \mathbb{R}$  and  $\alpha$  is basic, transversally harmonic (see [To] for the theory of basic Laplacian and basic cohomology etc.).

Let now  $S^1$  denote the unique homology class in  $H_1(M)$  (call it *the fundamental circle of  $\theta$* ) such that  $\int_{S^1} \theta = 1$  and  $\int_{S^1} \alpha = 0$  for every basic cohomology class  $\alpha$ .

As any complex submanifold of a compact Vaisman manifold is tangent to the Lee field and hence Vaisman itself,  $Y$  is Vaisman with Lee form  $\theta|_Y$ . Hence we deduce that the fundamental circle of  $\theta$  is the image of the fundamental circle of  $\theta|_Y$  under the natural map  $H_1(Y) \rightarrow H_1(M)$ .

We now integrate 3.1 on any  $\gamma \in H_1(Y)$  and take into account that  $\eta|_Y$  is exact to get  $c = 0$ . Hence,  $\eta$  basic. It can then be treated as a harmonic one-form on a Kähler manifold (or use the existence of a transversal  $dd^c$ -lemma). This implies  $d^c \eta = 0$ .

But then one obtains a contradiction, as follows. Letting  $J$  to be the almost complex structure of  $M$ , we see on one hand we have

$$\int_M d(\Xi^{n-1}) \wedge J(\theta) = \int_M (n-1) \Xi^{n-1} \wedge \theta \wedge J(\theta) > 0$$

since  $\Xi$  is positive. On the other hand, since  $d(J(\theta)) = 0$ , it follows that  $d(\Xi^{n-1}) \wedge J(\theta)$  is exact so  $\int_M d(\Xi^{n-1}) \wedge J(\theta) = 0$ , a contradiction. ■

The following result is certainly well-known, but since we were not able to find out an exact reference we include a proof here.

**Lemma 3.4:** Assume  $(U, g)$  is a Kähler complex manifold,  $Y \subset U$  a compact submanifold and let  $c : \tilde{U} \rightarrow U$  be the blow-up of  $U$  along  $Y$ . Then, for any open neighbourhood  $V \supset Y$ , there is a Kähler metric  $\tilde{g}$  on  $\tilde{U}$  such that

$$\tilde{g}|_{\tilde{U} \setminus c^{-1}(V)} = c^*(g|_{U \setminus V})$$

**Proof.** (due to M. Păun; see also [Vu]).

1. There is a (non-singular) metric on  $\mathcal{O}_{\tilde{U}}(-D)$  (where  $D$  is the exceptional divisor of the blow-up) such that:

1.A. Its curvature is zero outside  $c^{-1}(V)$ , and

1.B. Its curvature is strictly positive at every point of  $D$  and in any direction tangent to  $D$ .

Indeed, if such a metric is found, everything follows, as the curvature of this metric plus a sufficiently large multiple of  $c^*(g)$  will be positive definite on  $\tilde{U}$ .

2. To finish the proof, we notice that the existence of a metric  $h$  with property 1.B is clear, due to the restriction of  $\mathcal{O}_{\tilde{U}}(-D)$  to  $D$ .

Now let  $\alpha$  be its curvature; then  $\alpha - i\partial\bar{\partial}\tau = -[D]$  for some function  $\tau$ , with at most logarithmic poles along  $D$ , bounded from above, and non-singular on  $\tilde{U} \setminus D$ . Consider the function  $\tau_0 := \max(\tau, -C)$  where  $C$  is some positive constant, big enough such that on  $\tilde{U} \setminus c^{-1}(V)$  we have  $\tau > -C$ . Clearly, on a (possibly smaller) neighbourhood of  $D$  we will have  $\tau_0 = -C$ , such that the new metric  $e^{-\tau_0}h$  on  $\mathcal{O}_{\tilde{U}}(-D)$  also satisfies 1.A. ■

Now we can prove Theorem 2.8. Let  $c : \tilde{M} \rightarrow M$  be the blow up of  $M$  along the submanifold  $Y$ . Let  $g$  be a LCK metric on  $M$  and let  $\theta$  be its Lee form. Since  $Y$  is IGCK we see  $\theta|_Y$  is exact. Let  $U$  be a neighbourhood of  $Y$  such that the inclusion  $Y \hookrightarrow U$  induces an isomorphism of the first cohomology. Then  $\theta|_U$  is also exact, so, after possibly conformally rescaling  $g$ , we may assume  $\theta|_U = 0$  and hence  $g|_U$  is Kähler. In particular,  $\text{supp}(\theta) \cap U = \emptyset$ . Now choose a smaller neighbourhood  $V$  of  $Y$  and apply Lemma 3.4. We get a Kähler metric  $\tilde{g}$  on  $\tilde{U}$  which equals  $c^*(g)$  outside  $c^{-1}(V)$ , so it glues to  $c^*(g)$  giving a LCK metric on  $\tilde{M}$ . ■

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